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Complexity theory for the global optimizer

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What is Global Optimization?

Several interpretations:

- Many trials increase the chance of catching the global optimum
  - Multistart, scatter search
  - Metaheuristics: genetic algorithms, simulated annealing, particle swarm, ant colony...

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  - No way to know when it happens
- A method classification by Neumaier (2004)
  - **Incomplete**: clever heuristics
  - **Asymptotically complete**: converges eventually
  - **Complete**: converges and knows when prescribed tolerance is reached
  - **Rigorous**: converges despite rounding errors (floating point arithmetic)
- "Deterministic global optimization"

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- Results are sometimes negative
  - $n \times n$ chess/checkers/go cannot be solved in polynomial time
  - Conjecture A implies that problem B is C-hard
- The terminology is well-suited for studying global optimization (in the strict sense)

I will discuss some concepts in complexity theory and their implications for our field

- The NP class, approximation complexity, randomized algorithms

Algorithm Analysis

- First contact with runtimes
- Big O notation
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- Sorting algorithms
  - Naïve sort: $O(n^2)$ operations
  - Quicksort (1960): $O(n \log n)$ operations on average
  - Merge sort (1945): $O(n \log n)$ operations in worst case

Improved Runtimes

- Asymptotic improvements may have practical significance
  - Fast Fourier transform, $O(n \log n)$
  - Matrix multiplication
    - Trivial bounds: $O(n^3), \Omega(n^2)$
    - Strassen algorithm (1969): $O(n^{2.81})$

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  - Matrix multiplication
    - Trivial bounds: $O(n^3), \Omega(n^2)$
    - Strassen algorithm (1969): $O(n^{2.81})$
- ...or not
  - Coppersmith-Winograd algorithm (1987), $O(n^{2.38})$ more efficient only for enormous matrices

Tractability

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- Polynomial on what machine?
  - Any classical computer
  - Turing machines, random-access machines and other theoretical machines can simulate each other with only polynomial slow-down
  - Not true for quantum computers as far as we know

Example: Satisfiability

Does any truth assignment of the Boolean variables x, y, z satisfy the expression:

$$(x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee z) \wedge (x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y \vee \bar{z})$$

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- A satisfiability problem, 3-SAT
- Naïve algorithm: $O(2^n)$
- Best known: $O(1.439^n)$

“Easy to verify”

- Many difficult problems have this in common:
  - A ′yes′ answer can be verified quickly by checking a candidate (proof/certificate/witness)
- Define the class NP of problems for which any ′yes′ instance has a proof which can be checked in polynomial time

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- Examples
  - Decision versions of many optimization problems, discrete and continuous
  - Is $\min_{x \in X} f(x) < M$? If it is, then a point $x_0 \in X, f(x_0) < M$ is a proof
  - Graph isomorphism, traveling salesman, quadratic assignment, longest path, bin packing, knapsack, ...

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  - Games like Battleships, Mastermind, Minesweeper, ...

Complete Problems

Some problems in NP are *as hard as any other* NP *problem*

- If one of these problems can be solved in polynomial time, then so can any NP problem
- Cook and Levin proved that any NP problem can be *reduced* to (reformulated as) a satisfiability problem
- Karp (1972) listed 21 problems with the property

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- Karp (1972) listed 21 problems with the property
- Definition: C belongs to the class of NP-*hard* problems if
  - every problem in NP can be reduced to C in polynomial time
- Definition: C belongs to the class of NP-*complete* problems if
  - C is NP-hard, and
  - $C \in$ NP

Proving NP-hardness

If B is NP-complete and B reduces to C (in polynomial time), then C is NP-hard

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- Three common techniques: restriction, local replacement, component design
- Restriction: 0-1 Linear Programming is NP-complete
  $\Rightarrow$ MINLP is NP-hard
- Local replacement: Satisfiability to 3-SAT
  - A local replacement of long clauses by clauses with three literals

$$x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5$$
$$\downarrow$$
$$(x_1 \vee x_2 \vee y_1) \wedge (\bar{y_1} \vee x_3 \vee y_2) \wedge (\bar{y_2} \vee x_4 \vee x_5)$$

A recreational example: Lemmings is NP-complete (Cormode 2004)

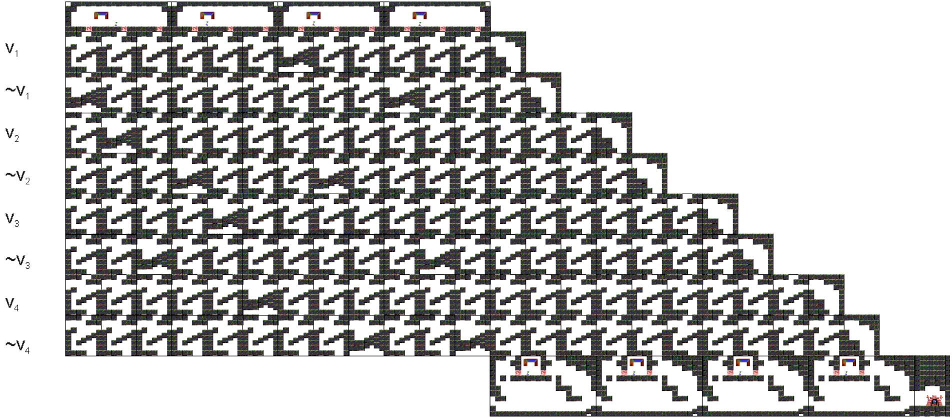

Figure 5: Lemmings level encoding the formula $(\bar{v_1} \vee v_2 \vee \bar{v_3}) \wedge (\bar{v_2} \vee v_3 \vee v_4) \wedge (v_1 \vee \bar{v_2} \vee \bar{v_4}) \wedge (\bar{v_1} \vee \bar{v_3} \vee \bar{v_4})$

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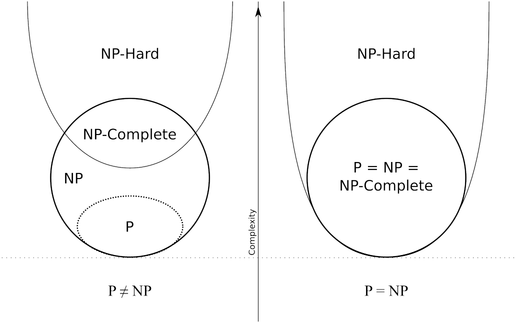
Is P ≠ NP?

- Maybe all problems in NP have an efficient algorithm?
- Why didn't we find one yet?
- Often stated as "P versus NP", one of the Millennium Prize problems



Decision Problems and Optimization

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  What is the minimum value of $f(x), x \in X$?
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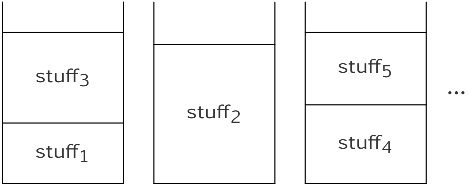
- The optimization version is at least as hard as the decision version
- In practice we rarely need the exact optimum
- Is approximation any easier?

Approximative Solutions

A solution x to a minimization problem is ε-optimal if:

$$f(x) \leq (1+\varepsilon)\, f(x^*)$$

- Example: Bin packing, first-fit



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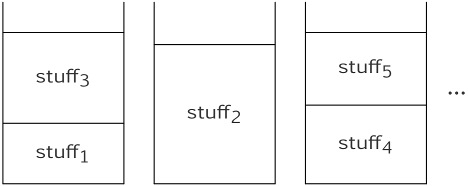

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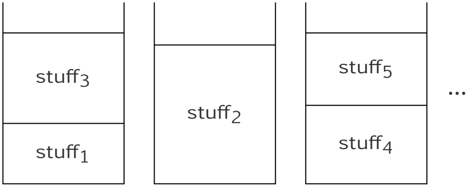

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- $N_{FF} < 2N^*$ (an improved analysis gives $N_{FF} \leq 1.7N^* + 2$)

Approximation Complexity

By reducing approximation problems to NP-complete decision problems they are shown to be hard

- The gap technique
  - Objective range $\subset (0, a] \cup [b, +\infty)$
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- Many hardness results followed on the PCP Theorem (Arora et al. 1990)
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- A hierarchy of complexity classes emerges: APX $\supset$ PTAS $\supset$ FPTAS
- The classes are not equal unless P $=$ NP

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- $\notin$ APX: Linear Integer Programming, general TSP, and Quadratic Assignment have no efficient approximation algorithms for any constant ε (unless $\mathrm{P} = \mathrm{NP}$)

Polynomial Time Approximation Schemes

PTAS: problems with polynomial time approximation algorithms for any ε

- Geometric Traveling Salesperson
  - Euclidean distances or $l_p, p \geq 1$ norm
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- Grows exponentially with $1/\varepsilon$

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- Knapsack Problem (see Vazirani 2001)
  - $O(n^3/\varepsilon)$
- Quadratic Programming
  - Compact polytope, t negative eigenvalues
  - L = complexity of solving a convex QP of the same size
  - (Vavasis 1992)
$$O\left(\left\lceil n(n+1)/\sqrt{\varepsilon}\right\rceil^t L\right)$$
  - Fully polynomial if t is bounded

So which Problems are Hard?

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- Plenty of references
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- Continuous problems: "convex easy, nonconvex hard"
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  - Self-concordant barrier functions exist for all closed convex solids
- Some exceptions
  - Geometric programming: posynomials $cx_1^{p_1} \cdots x_n^{p_n}, c > 0$
  - Linear fractional programming: $(p^T x + \alpha)/(q^T x + \beta)$
  - ...

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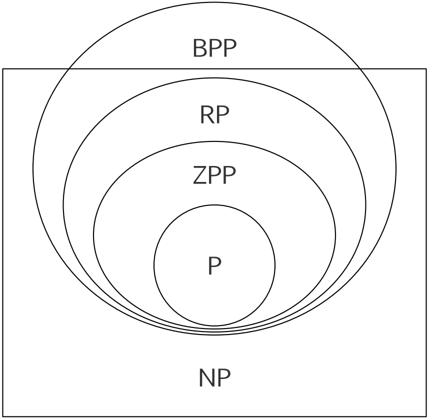
- BPP - bounded-error probabilistic polynomial-time
  - wrong answer with probability $\leq 1/3$
- RP - randomized polynomial-time
  - outputs ′no′ if the correct answer is ′no′
  - outputs ′no′ if the correct answer is ′yes′ with probability $\leq 1/2$

Randomized Decision Classes



Example: MAX-3-SAT

$$\bigwedge_{k=1}^{K} (x_{k_1} \vee x_{k_2} \vee x_{k_3})$$

- A clause (three different literals) is satisfied by a random assignment with probability

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- Can be derandomized to give deterministic algorithm

Example: Primality Testing

Is n a prime number?

- Let $D = \log n$, the number of digits in n
- Adleman-Pomerance-Rumely (Jacobi sums), $O(D^{c \log\log D})$
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- Elliptic Curve Primality Proving
  - Expected runtime $\tilde{O}(D^4)$
  - Primality testing $\in$ ZPP
- Agrawal-Kayal-Saxena (2002), $\tilde{O}(D^6)$
  - Deterministic
  - Primality testing $\in$ P

Primality testing was known to be in BPP, and now in P

- It has been conjectured that $P = BPP$
- Randomized algorithms might not be fundamentally stronger
- But they may have advantages
  - Lower degree runtimes
  - Sometimes conceptually easier, faster to program
- A probability of errors may be tolerable if it can be bounded
  - Example: “industrial strength primes”

References

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Questions?